Three dimensional massive scalar field theory and the derivative expansion of the renormalization group

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Abstract

We show that non-perturbative fixed points of the exact renormalization group, their perturbations and corresponding massive field theories can all be determined directly in the continuum – without using bare actions or any tuning procedure. As an example, we estimate the universal couplings of the non-perturbative three-dimensional one-component massive scalar field theory in the Ising model universality class, by using a derivative expansion (and no other approximation). These are compared to the recent results from other methods. At order derivative-squared approximation, the four-point coupling at zero momentum is better determined by other methods, but factoring this out appropriately, all our other results are in very close agreement with the most powerful of these methods. In addition we provide for the first time, estimates of the n-point couplings at zero momentum, with $n = 12, 14$, and the order momentum-squared parts with $n = 2 \cdots 10$.

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1. Introduction

The motivations for the present paper are three-fold. Firstly, we wish to expand upon the renormalization group reasons, given in ref.[1], for the quantization of renormalised couplings. Since these couplings correspond to relevant (and marginally relevant) perturbations about a fixed point, this necessitates first briefly reviewing how the renormalization group also 'self-determines' the fixed point structure itself [2] –[5]: as we recently emphasised[5], the fixed-point and its eigen-operator spectrum – and hence the corresponding massless quantum field theory, can be deduced and computed entirely from consistency arguments applied to the effective action and its exact renormalization group flow close to the fixed point – *i.e.* without needing to go through the construction of introducing an overall cutoff Λ_0, a sufficiently general bare action, and then taking the continuum limit $\Lambda_0 \to 0$. Thus the universal continuum properties are accessed directly without this standard, but for quantum field theory, actually artificial and extraneous, scaffolding.

It would seem rather strange if having been able in this way to derive directly the continuum massless theory, the corresponding massive theory required the reintroduction of the concepts of bare couplings and corresponding tunings to reach the continuum limit. At first sight this appears to be the case, however the second and main purpose of the present paper is to show that once the massless theory's fixed point and eigen-operator spectrum are known, the massive theory may be constructed, again directly, without recourse to any limiting procedure. An important byproduct of this analysis, is the transformation of the continuum limit of the Wilsonian RG (Renormalization Group) into the form of a self-similar flow for the underlying relevant couplings, in close analogy to the usual field theory perturbative RG, although here the β functions are defined non-perturbatively.[1]

We concentrate on a description within the derivative expansion approximation to the renormalization group[2] (see also [9]–[11]), but point out at the appropriate points how and why precisely the same effects can be expected to work in determining the fixed points, eigen-operator spectrum and massive continuum limits also for the exact renormalization group.

Finally, we apply these concepts to a calculation, which however is interesting in its own right: the universal coupling constant ratios of the three dimensional Ising universality class scaling equation of state. In quantum field theory terms, this corresponds to determining the one-particle irreducible n-point functions at zero momentum (equivalently the $O(p^0)$

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[1] Some related remarks can be found in refs.[6]–[8].

terms in a series expansion in powers of the momenta) for the three-dimensional one-component massive scalar field theory based about the non-perturbative Wilson-Fisher fixed point. (These are universal once they are divided by appropriate powers of the scalar's mass.) We compute these universal ratios both for the Local Potential Approximation of the Wegner-Houghton equation[12][13], and for a smooth cutoff as utilised in refs.[2][9]. The latter allows also to go to next order – $O(\partial^2)$ – in the derivative expansion. We will see that the results improve, and compare well with the very recent high order perturbation theory results of Guida and Zinn-Justin[14]. Indeed, appropriate universal higher coupling constant ratios lie within the errors of the corresponding resummed perturbation theory results[14]. Moreover, we are able to give estimates for even higher point couplings ($n = 12 \cdots 14$), and also to estimate the corresponding universal $O(p^2)$ terms in the first six n-point functions. At present, these are not available from any other method.

Computing the n-point couplings of the scaling equation of state in this way is actually a slightly eccentric use of the derivative expansion approximation, since the results most naturally present themselves as full numerical solutions for the equation of state. Indeed in ref.[15], a numerical computation of the scaling equation of state[2] has already been presented. These authors obtain the equation of state by tuning an appropriate bare action, and solving the RG within a modified $O(\partial^0)$ approximation incorporating anomalous scaling (*viz.* η). Our purpose for computing instead the corresponding Taylor expansion coefficients, was primarily to provide a direct comparison of the first two orders of the derivative expansion *per se* with the emerging accurate resummed perturbation theory results, preliminarily announced by Zinn-Justin, and some independently also by Sokolov[17] at the August conference "RG96" in Dubna. For this comparison, it is sufficient to compute only in the symmetric phase. (The conceptual points of the paper however apply to either phase.) A similar computation in the broken phase would be very interesting and allow estimation also of 'amplitude ratios'[15] by the derivative expansion. Alternatively these could be computed from the couplings obtained here, by analytic continuation in the mass[14]. Extensions to three dimensional $O(N)$ symmetric N-component scalar field theory, and to massive two dimensional theories, in particular the infinite sequence of multicritical fixed points, described by one-component scalar field theory, seems straightforward, the formalism to $O(\partial^2)$ for these cases having already been developed [5][9][18]. Also, it is certainly possible to estimate even higher point couplings, using a more serious

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^{2} See ref.[16] for a lattice field theory computation.

numerical attack than we attempt in this paper. (All the computations for this paper were performed within the Maple package.)

A more detailed précis of the present paper now follows. In sec.2, we set up some of the notation and recall some salient facts: the extraction of the Legendre effective potential (and action) from the $\Lambda \to 0$ limit of the Wilson effective action – which is a necessary step to construction of the equation of state from the exact RG, the self-determination of the fixed point structure through the requirement of non-singularity, the large field behaviour and thus determination of the fixed point's Legendre effective potential. We point out that the leading large field asymptotics are independent of the effective cutoff Λ, physically a reflection of the fact that quantum corrections to the free energy are negligible for sufficiently large fields.

In sec.3, we study the spectrum of perturbations about the fixed point (*a.k.a.* eigen-operators). We show that quantization of the spectrum of dimensions of these operators follows from the requirement that the scale dependence can be absorbed in an associated coupling, *i.e.* a renormalised coupling, and thus correspond to the universal self-similar flow characteristic of the continuum limit. This is not guaranteed beyond the linearised level. By examining the flow exactly, which can be done for large fields, we establish that only perturbations behaving as a power of ϕ for large field ϕ, can be associated with a renormalised coupling. As a byproduct we derive the form of the Legendre effective potential to first order in the associated physical coupling. While these results are established within the Local Potential Approximation (LPA), we explain why (as in sec.1) the same results should follow also for the exact RG. Finally in this section, we apply Sturm-Liouville theory to the LPA to prove that those perturbations that do not behave as power-law for large ϕ, collapse under evolution with scale into an infinite sum of perturbations that do behave as power-law. Thus all continuum physics is described in terms of the fixed point, the 'power-law perturbations', and their associated renormalised couplings. The main statements of this section confirm and generalise the corresponding findings of ref.[1].

The previous two sections thus describe how to obtain *directly* any massless continuum limit, and its associated operator spectrum, "directly" because no limit is actually taken. In sec.4, we determine how to set up directly the corresponding massive continuum limits. We first review the standard lore on how a non-perturbative massive continuum limit is obtained, defining in the process the so-called renormalised trajectory. We show that this trajectory can be defined directly in terms of a boundary condition on the flow, involving a coupling g, such that finite values of g yield finite continuum limits, and

obtain as a consequence the leading terms in large ϕ dependence of the corresponding massive continuum limit. The boundary condition still involves a limiting procedure (its dependence on Λ is prescribed as $\Lambda \to \infty$), but we show that this limit can also be removed by reexpressing the flow in terms of the underlying renormalised couplings and their associated beta-functions. In addition we set up a change of variables so that the universal ratios of couplings (*viz.* physical scale independent ratios) are obtained once all modes have been integrated out (*i.e.* as $\Lambda \to 0$). Again in this section, we point out how and why these observations should hold for the exact RG.

In sec.5, we formulate the theory of sec.4 specifically in terms of the LPA of the sharp cutoff (*a.k.a.* Wegner-Houghton) flow equation, applied to three dimensional one-component scalar field theory governed by the non-perturbative Wilson-Fisher fixed point, particularly the universal coupling constant ratios of the Legendre effective potential (equivalently scaling equation of state). We discuss the numerical implementation of the boundary conditions and flows, expansion directly in terms of the (ratios of) couplings, and the corresponding choice of (and independence of) closure ansatz. The short sec.6 delineates the differences that arise in using the $O(\partial^0)$ approximation, equivalently LPA, of the smooth cutoff flow equation derived in ref.[2].

In sec.7 we go beyond the LPA, to order derivative-squared, thereby also allowing estimation of universal ratios of the momentum-squared terms in n-point Green functions, and more accurate estimation of the scaling equation of state. Wave-function renormalisation and running anomalous dimensions are dealt with explicitly here. Once again we provide here also the explicit formulae needed for the numerics.

Finally, in sec.8 we present the numerical results. We present numerical evidence showing independence of closure ansatz, for three different choices of ansatz. Picking the most rapidly converging ansatz, we deduce the sharp cutoff LPA, the $O(\partial^0)$, and $O(\partial^2)$ results for the universal coupling constant ratios, and combine the derivative expansion results into one final number together with an estimated error of truncation of the derivative expansion. These are compared to results from resummed perturbation theory[14]. The perturbative methods are more powerful for low order couplings but the derivative expansion eventually wins out for higher order couplings. Following ref.[14], we then factor out an overall normalization (effectively the size of the four-point coupling) and present derivative expansion results (and sharp cutoff LPA results) for the corresponding ratios F_{2l-1}. These are seen to be much more accurately determined by the derivative expansion, and are in close agreement with the most accurate determinations from other methods.

(A table of comparisons is presented, including the results from resummed perturbation theory, ε expansion, the exact RG approximation of ref.[15], high temperature series and Monte-Carlo estimates.) We thus conclude that the main error in the derivative expansion is in determining the lowest point coupling(s), and can be absorbed in an effective normalisation factor ζ_{eff}.

There is no need for a separate summary and conclusions, since this has already in effect been incorporated in the above introduction and précis.

2. Fixed Points

First, let us review how the fixed points are determined by consistency arguments alone. To do this, we find it easier to concentrate on a specific equation and then to point out how the arguments generalise. Thus consider the so-called LPA (Local Potential Approximation) [13] to the Wegner-Houghton renormalization group[12]. We remind the reader that the latter is just[2][6] the sharp cutoff[3] limit of Polchinski's form[19] of Wilson's continuous RG[20] (Renormalization Group):

$$\frac{\partial S_\Lambda}{\partial \Lambda} = \frac{1}{2} \operatorname{tr} \frac{\partial \Delta_{UV}}{\partial \Lambda} \left\{ \frac{\delta S_\Lambda}{\delta \phi} \frac{\delta S_\Lambda}{\delta \phi} - \frac{\delta^2 S_\Lambda}{\delta \phi \delta \phi} - 2 \left(\Delta_{UV}^{-1} \phi \right) \frac{\delta S_\Lambda}{\delta \phi} \right\} \quad ,$$

(where $\Delta_{UV}(q, \Lambda) = C_{UV}/q^2$ is a cutoff massless propagator. For more details on this equation see refs.[5][6][19][20]. It will not however be used in the rest of the paper.) The approximation corresponds to projecting on an effective action of the form

$$S_\Lambda = \int d^D x \left\{ \tfrac{1}{2} (\partial_\mu \phi_{\mathrm{phys}})^2 + V_{\mathrm{phys}}(\phi_{\mathrm{phys}}, \Lambda) \right\} \quad , \tag{2.1}$$

i.e. to discarding all momentum dependent terms generated by the RG. (D is the space-time dimension.) For a single component scalar field, the result is[13][21][22][3][11]:

$$\frac{\partial}{\partial t} V(\phi, t) + d\phi V'(\phi, t) - DV(\phi, t) = \ln [1 + V''(\phi, t)] \quad . \tag{2.2}$$

Here $t = \ln(\mu_p/\Lambda)$, μ_p is an arbitrary physical mass scale, the primes stand for differentials with respect to ϕ, and d is the scaling dimension of the field ϕ. Generally this is given at a

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[3] We look first at this long established version of the LPA. Later we will employ smooth cutoff versions, which also turn out to be more accurate. The subtleties involved in handling a sharp cutoff, both technical and physical, have been extensively addressed in ref.[11].

fixed point by $d = \frac{1}{2}(D - 2 + \eta)$, however a consequence of the LPA is that the anomalous dimension $\eta = 0$. In this equation the scaling dimensions of the field and potential have been used to construct dimensionless equivalents [3]:

$$\phi = \phi_p/\Lambda^d \qquad \text{and} \qquad V(\phi, t) = V_p(\phi_p, \Lambda)/\Lambda^D \quad . \tag{2.3}$$

Thus p-subscripted quantities refer to dimensionful *physical* quantities. This step is equivalent to (but more direct than) the traditional rescaling of the cutoff back to its original size after a blocking[20][12].

Actually, a further purely numerical transformation is necessary to obtain the properly normalised physical quantities appearing in (2.1):

$$\phi_p = \zeta \phi_{\text{phys}} \qquad \text{and} \qquad V_p = \zeta^2 V_{\text{phys}} \quad , \tag{2.4}$$

where $\zeta = (4\pi)^{D/4}\sqrt{\Gamma(D/2)}$ was chosen to prettify the eqn.(2.2). This step will only need to be taken into account in the final sections containing the numerics.

It is worth recalling here that the effective potential of the Wilsonian effective action S_Λ tends, in the limit $\Lambda \to 0$, to the Legendre effective potential[6][4] (*i.e.* generator of one-particle irreducible Green functions evaluated at zero momentum). The latter is of course precisely what we will need later, to construct the equation of state. This is one consequence of the deeper connection also proved in ref.[6], namely that the Legendre effective action (minus its infrared regulated kinetic term[6]): Γ_Λ, for a field theory with infrared cutoff $C_{IR}(q, \Lambda)$ imposed, is a generalised Legendre transform of Polchinski's Wilsonian effective action S_Λ computed with ultra-violet cutoff $C_{UV}(q, \Lambda) = 1 - C_{IR}(q, \Lambda)$: [5]

$$S_\Lambda[\phi] = \Gamma_\Lambda[\phi^c] + \tfrac{1}{2}(\phi^c - \phi).\frac{q^2}{C_{IR}}.(\phi^c - \phi) \quad . \tag{2.5}$$

(Somewhat similar remarks have been made by other authors[24][25]. *N.B.* ϕ^c in this equation is the "classical field", defined as a function of ϕ through the Legendre transform, in the usual way, *i.e.* by differentiating (2.5) by ϕ at fixed ϕ^c.) In this way we can have our cake and eat it: fixed points in the Wilsonian RG have a natural physical explanation through thinking about blocking, but at the same time we can recover the information apparently blocked away – via this generalised Legendre transform relation.

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[4] providing that $\frac{\partial}{\partial q^2}C_{UV}|_{q=0} = 0$, as follows trivially from observations in refs.[6][11][23]

[5] It does not matter here whether quantities are physical, or dimensionless combinations.

At a fixed point, $V(\phi, t) = V(\phi)$ is fixed *i.e.* independent of t. From eqn.(2.2), it would appear at first glance that there is a two parameter set of fixed point solutions $V(\phi)$ (because (2.2) reduces to a second order ordinary differential equation), but this is true only locally. In fact all but typically a discrete number[6] of these solutions are singular at some finite value of the field ϕ [2][9][1][5]. Clearly the presence of such a singularity is unacceptable *e.g.* in leading to violations of Griffiths' analyticity. From our experience, the solutions that are well-defined for all real ϕ (typically then countable in number), may all be identified with approximations to the expected fixed points of the exact RG, in the sense that they have the right qualitative properties and yield reasonable quantitative answers [2][9][1][5]. Importantly there appear to be none of the 'spurious fixed points' that afflict higher order truncations in powers of the field[3][5].

The form of the fixed point solution for large ϕ may be ascertained from eqn.(2.2) [3]:

$$V(\phi) \sim A\phi^{1+\delta} \qquad \text{as} \qquad \phi \to \infty \quad , \tag{2.6}$$

with $A \geq 0$ an as yet unkown coefficient. We have used a hyperscaling relation to write $D/d - 1$ as the critical exponent δ. (We are assuming $d > 0$ here and from now on. This thus excludes the two-dimensional exceptions[9]. All scaling relations trivially hold automatically in derivative expansion approximations since scale invariance is not broken and the massless limit may be directly considered, as we are indeed so doing here.) The leading term above remains valid beyond the LPA and for any cutoff *in the form written*[7] because it simply arises from solving the left hand side of eq.(2.2),[8] which in turn is purely generated by the assignment of scaling dimensions d and D to the field and potential respectively. This power law is thus precisely as required so that $V_p(\phi_p, \Lambda)$ has a finite non-vanishing limit as $\Lambda \to 0$. Indeed, from (2.3) and (2.6), we obtain

$$V_p(\phi_p, 0) = A\phi_p^{1+\delta} \quad . \tag{2.7}$$

In view of the relation to the Legendre effective potential outlined above, this is nothing but the free energy per unit volume, at the critical point, as a function of the field [3]. Notice that the behaviour of $V_p(\phi_p, \Lambda)$ in the regime $\phi_p \gg \Lambda^d$, is also given by (2.7), and in particular is thus actually independent of Λ. Physically, this is precisely to be

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[6] an exception being critical sine-Gordon models in two dimensions[9]

[7] *i.e.* with $\delta = D/d - 1$, and d the full scaling dimension following from a non-zero η.

[8] the neglect of the right hand side being justified by inspection

expected: it reflects the fact that quantum corrections to the free energy are negligible for sufficiently large fields. Indeed the asymptotic behaviour (2.6) for a fixed point, is the only such behaviour consistent with this 'mean-field-like' evolution *i.e.* one in which quantum corrections are absent.

It seems reasonable to suppose that the same characteristics, as outlined above, also completely determine the fixed points in the exact renormalisation group. In this case[6], the fixed point equation is a second order non-linear functional differential equation and presumably, locally has a full functional space worth of solutions but globally only typically a discrete number which are well defined for all $\phi(\mathbf{x})$, these latter being again the only *bona fide* fixed points.

3. The eigen-operators

To obtain the spectrum of (integrated scalar) eigen-operators we linearize around the fixed point by writing $S_\Lambda[\phi] = S[\phi] + \varepsilon\, e^{\lambda t} s[\phi]$, or in terms of the LPA:

$$V(\phi, t) = V(\phi) + \varepsilon\, e^{\lambda t} v(\phi) \quad , \tag{3.1}$$

to first order in ε. Here we have used the fact that the RG is quasilinear in t, and used separation of variables. Thus we have from (2.2):

$$\lambda v(\phi) + d\phi v'(\phi) - Dv(\phi) = \frac{v''(\phi)}{1 + V''(\phi)} \quad . \tag{3.2}$$

This time the ordinary second order differential equation is linear with non-singular co-efficients and therefore we are guaranteed to have for *every real value of λ*, a continuous one-parameter (up to the arbitrary normalisation) set of *globally well-defined* real solutions. How can we square this with the fact that experiment, simulation *etc.* , typically only uncover a discrete spectrum of such operators and in particular only a discrete number of relevant directions corresponding to eigenvalues $\lambda > 0$? The answer was given in ref.[1][5](if rather compactly): *only the discrete set of solutions for λ and $v(\phi)$, where $v(\phi)$ behaves as a power of ϕ for large field, can be associated with a corresponding renormalised coupling $g(t)$ and thus universal self-similar flow close to the fixed point.*

This comes about as follows. First note that if $v(\phi)$ is to behave as power-law for large ϕ, then from (3.2), this power is determined as

$$v(\phi) \sim a\phi^{1 + \delta - \lambda/d} \qquad \text{as} \qquad \phi \to \infty \quad , \tag{3.3}$$

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with a an as yet unkown coefficient. Again, as with (2.6), this remains valid beyond the LPA and for any cutoff since the power is determined only by the left hand side of (3.2). Once this power law is imposed for $\phi \to \infty$, and for $\phi \to -\infty$ with possibly different coefficient,[9] and linearity of (3.2) taken into account, we see that we have sufficient boundary conditions to overconstrain (3.2) leading to typically a discrete set of solutions for the eigen-values λ and eigen-operators $v(\phi)$ [2]. Therefore we need now to explain only why we require $v(\phi)$ to behave as a power-law for large ϕ.

Studying eqn.(3.2), and using (2.6), we see that if $v(\phi)$ does not behave as (3.3) then for $A > 0$, it must behave as

$$ v(\phi) \sim \exp A(D - d)\phi^{1+\delta} \qquad \text{as} \qquad \phi \to \infty \quad , \tag{3.4} $$

where we are neglecting a multiplicative power-law correction. Only for the Gaussian fixed point $V(\phi) \equiv 0$, does $A = 0$, and in this case we have instead $v(\phi) \sim \exp d\phi^2/2$. The behaviour (3.4) is changed by going beyond the LPA, and depends on the choice of cutoff[10] because it results from balancing the right hand side of (3.2) against the derivative term of the left hand side. The crucial point for us however is that if $v(\phi)$ does not behave as (3.3), then it diverges faster than power-law as $\phi \to \infty$. And this statement is presumably universally true, holding for any (valid) choice of cutoff and also for the exact RG.

From eqn.(3.1), it is tempting to identify $g(t) = \varepsilon e^{\lambda t}$ as a (renormalised) coupling conjugate to the operator $v(\phi)$, but this t evolution followed from linearising in ε, and for any finite coupling, no matter how small, there is a regime of large fields where such linearisation is not justified. [Thus *e.g.* $\varepsilon v(\phi) \gg 1$ for $\phi \gg (-\ln \varepsilon)^{d/D}$ in case (3.4).] In this regime however, it is straightforward *to solve exactly* for the t evolution because the right hand side of (2.2) contributes negligably compared to the at-least-power-law behaviour of the left hand side, while the left hand side –being linear– is easily solved. Thus starting at $t = 0$ with $V(\phi, 0) = V(\phi) + \varepsilon v(\phi)$, we have

$$ V(\phi, t) \sim e^{Dt} V(\phi e^{-dt}, 0) \quad = V(\phi) + \varepsilon e^{Dt} v(\phi e^{-dt}), \quad \text{as} \quad \phi \to \infty \quad , \tag{3.5} $$

where we have used (2.6) (and are assuming, here and later, that subleading corrections to each term can be neglected). Indeed, once again, the neglect of the right hand side of

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[9]   or a boundary condition following from symmetry considerations, is imposed at the origin[9][5]

[10]   *e.g.* compare the formulae of ref.[2].

(2.2) is just the statement that quantum corrections are negligible in the regime of large fields; combining (3.5) and (2.3),

$$V_p(\phi_p, \Lambda) \sim \mu_p^D V(\phi_p/\mu_p^d, 0) \qquad \text{for} \qquad \phi_p \gg \Lambda^d \quad . \tag{3.6}$$

We expect then that (3.5) and (3.6) hold also for the exact RG. Now, for the power-law perturbations (3.3), (3.5) simplifies to

$$V(\phi, t) \sim A\phi^{1+\delta} + \varepsilon e^{\lambda t} a \phi^{1+\delta-\lambda/d} \qquad \text{as} \qquad \phi \to \infty \quad , \tag{3.7}$$

which is just the large ϕ behaviour of (3.1) – confirming the identification $g(t) = \varepsilon e^{\lambda t}$. [11] Correspondingly from (3.6):

$$V_p(\phi_p, 0) = A\phi_p^{1+\delta} + \varepsilon \mu_p^\lambda a \, \phi_p^{1+\delta-\lambda/d} \quad , \tag{3.8}$$

from which we see that $g(t)$ corresponds to a dimensionful μ_p-dependent physical coupling $g_p = \varepsilon \mu_p^\lambda$ of dimension λ. But for the non-power-law perturbations (3.4), and in general for perturbations rising faster than power-law, the t dependence in (3.5) cannot be combined with ε and thus absorbed into a running coupling. (Nor can all the μ_p dependence be absorbed in a physical coupling.) We have thus confirmed the italicized statement at the beginning of this section, and expect that the statement holds beyond the LPA and indeed also for the exact RG.

Finally, we know from Sturm-Liouville theory[26] applied to (3.2) that *as soon as $t > 0$, non-power-law perturbations [i.e. behaving as (3.4), (3.5)], can be reexpanded in terms of a sum over the power-law perturbations. It follows then that in any case only these are needed to span all the continuum physics.*

To prove these statements, we note that the differential operator $\mathcal{D}$, defined by writing (3.2) as $\mathcal{D}v(\phi) = \lambda v(\phi)$, is Hermitian with respect to the weight function

$$\rho(\phi) = \{1 + V''(\phi)\} \exp - d \int^\phi d\tilde{\phi} \, \tilde{\phi} \left\{ 1 + V''(\tilde{\phi}) \right\} \quad , \tag{3.9}$$

and the power-law boundary conditions (3.3) (imposed at $\phi = \pm\infty$). This follows because the large ϕ behaviour of ρ is sufficient to ensure that the bilinear concomitant (the boundary term generated by integration by parts) vanishes. Indeed from (2.6),

$$\rho(\phi) \sim \exp - A(D - d)\phi^{1+\delta} \qquad \text{as} \qquad \phi \to \infty \quad , \tag{3.10}$$

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[11] We checked that at subleading order in the asymptotic expansion, the t dependence can still be entirely absorbed in $g(t)$.

for $A > 0$, otherwise $A = 0$ and $\rho \sim \exp{-d\phi^2/2}$ (neglecting again, here and in the ensuing, the unimportant multiplicative power-law corrections). Now, from the general Sturm-Liouville theory, we have that the power law eigen-functions $v_i(\phi)$ form a discrete set, $i = 1, 2, \cdots$, with corresponding, possibly finitely degenerate, eigen-values λ_i, such that there is a most positive eigenvalue, which we call λ_1, and an infinite tower of negative eigenvalues (*i.e.* taking the λ_i's to be non-increasing in i, we have $\lambda_i \to -\infty$ as $i \to \infty$. Note that these facts concur with physics expectations). The eigen-functions may be taken to be ortho-normalised: $\int_{-\infty}^{\infty} d\phi \rho(\phi) \, v_i(\phi) v_j(\phi) = \delta_{ij}$. From (3.4) and below it, and (3.5), we have that a non-power-law perturbation $v(\phi, t) \equiv V(\phi, t) - V(\phi)$ behaves as

$$
\begin{aligned}
A > 0: &\qquad v(\phi, t) \sim \exp A(D - d) \, e^{-Dt} \phi^{1+\delta} \\
A = 0: &\qquad\qquad \sim \exp d \, e^{-2dt} \phi^2/2
\end{aligned}
\qquad \text{as} \qquad \phi \to \infty \quad,
$$

thus the expansion coefficients $c_i(t) = \int_{-\infty}^{\infty} d\phi \rho(\phi) \, v(\phi, t) v_i(\phi)$, are well defined for all $t > 0$, while for $t > \frac{\ln 2}{D}$ ($A > 0$, or $t > \frac{\ln 2}{2d}$ with $A = 0$), we have the completeness relation,

$$
\int_{-\infty}^{\infty} d\phi \, \rho(\phi) \left\{ v(\phi, t) - \sum_{i=1}^{N} c_i(t) v_i(\phi) \right\}^2 \to 0 \qquad \text{as} \qquad N \to \infty \quad.
$$

We have thus proved the italicized statements of the previous paragraph. Note that these statements are the generalisation to any fixed point, of the corresponding statements made about Gaussian fixed points in ref.[1].

4. Massive continuum limits

From this detailed study we actually have in place everything we need to set up *directly* the massive continuum limit, *i.e.* without actually having to perform the limit. Before doing this however, let us recall the standard lore[20] on how a non-perturbative massive continuum limit is obtained. This is illustrated in fig.1. In the infinite dimensional space of bare actions, there is the so-called critical manifold, which consists of all bare actions yielding a given massless continuum limit. Any point on this manifold – *i.e.* any such bare action – flows under a given RG towards its fixed point; local to the fixed point, the critical manifold is spanned by the infinite set of irrelevant operators. Let us assume that emanating out of the fixed point there is just one relevant direction; the generalisation to more than one is straightforward and will be indicated later. Choosing an appropriate parametrization of the bare action, we move a little bit away from the critical manifold.
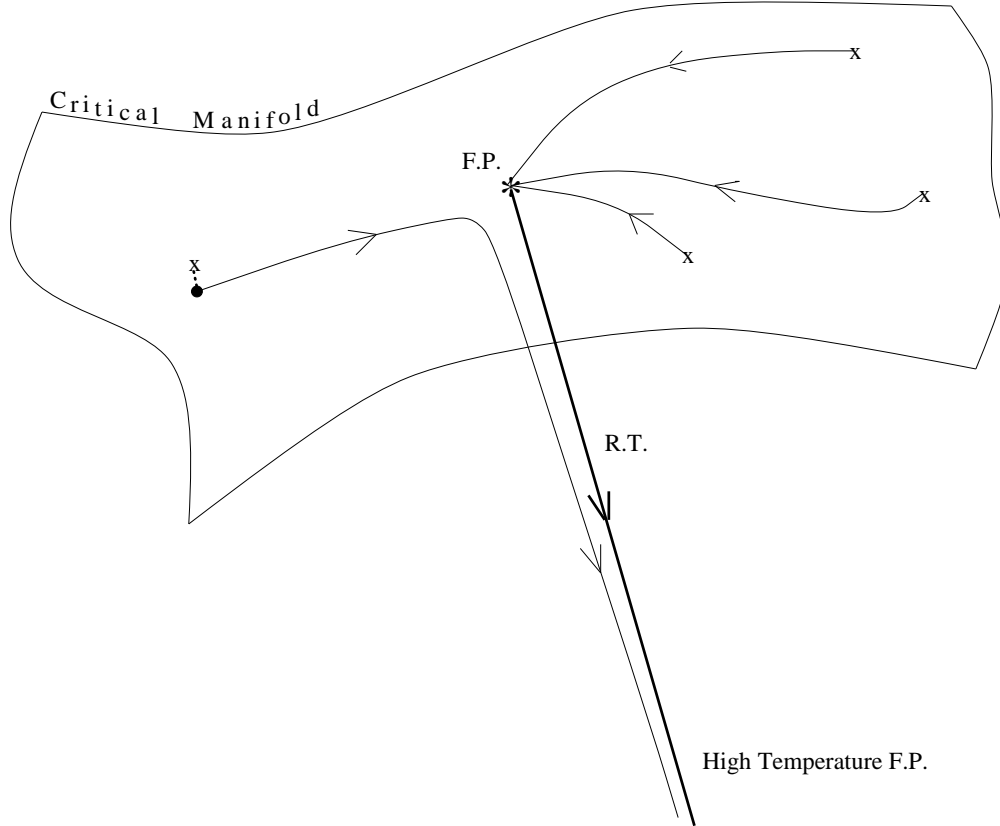
Fig.1. Tuning to a massive continuum limit. x marks points on the critical manifold, whereas the black blob is a point shifted slightly off the critical manifold.

The trajectory of the RG will to begin with, move towards the fixed point, but then shoot away in the relevant direction (exponentially fast in t) towards the so-called high temperature fixed point which represents an infinitely massive quantum field theory.

To obtain the continuum limit, and thus finite masses, one must now tune the bare action back towards the critical manifold and at the same time, reexpress late t physical quantities in renormalised terms appropriate for the diverging correlation length. In the limit that the bare action touches the critical manifold, the RG trajectory splits into two: a part that goes right into the fixed point, and a second part that emanates from the fixed point out along the relevant direction. This path is known as the Renormalised Trajectory[20] (RT); the effective actions on this path are 'perfect actions'[27]. In terms of renormalised quantities, the large t section of this path obtains a finite limit, namely the effective action of the continuum quantum field theory.

Therefore, to obtain the limits directly we must first describe the RT. This is given, *close* to the fixed point, by

$$V(\phi, t) = V(\phi) + g(t)v(\phi) \quad , \tag{4.1}$$

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where the relevant coupling $g(t)$, corresponding to the one relevant direction, satisfies

$$g(t) = g\mathrm{e}^{\lambda t} \quad , \tag{4.2}$$

with g a constant. Eqn.(4.1) is the unique solution that 'tracks back' into the fixed point as $t \to -\infty$. It satisfies the flow equation there [by linearisation as in (3.1)] since $g(t) \to 0$. Using (4.1) as a $t = -\infty$ 'boundary condition' for the flow equation, we may now describe the whole RT.

In a case where there is more than one relevant direction, we then of course have a continuum of possible RT's, which may be labelled by the ratios of the relevant couplings, thus labelling different physics for given overall mass-scale (*a.k.a.* correlation length). Eqn.(4.1) is replaced by a sum over the relevant couplings, each of which satisfies an equation of the form (4.2) as $t \to -\infty$. These generalizations are straightforward and thus will not be considered further.

Note now, that any finite value of g yields a finite limit for the continuum quantum field theory. To see this, note that since $\lambda > 0$, comparison of (3.3) with (2.6) shows that for *any* finite t, there is a sufficiently large ϕ where the linearised solution (4.1) is still valid! Indeed, the linearisation (3.2) is valid for (4.1), providing we have $g(t)v''(\phi)/V''(\phi) \ll 1$, and this is true for all $\phi \gg Cg^{\beta}\mathrm{e}^{dt}$. (Here $C = \frac{a(1+\delta-\lambda/d)(\delta-\lambda/d)}{A\delta(1+\delta)}$, and we have used the hyper-scaling relation $\beta = d\nu = d/\lambda$.) Taking the limits $t \to \infty$ and $\phi \to \infty$, while obeying the above inequality, we may continue to use (4.1) and (4.2) and thus, similarly to (3.7) and (3.8), we obtain the scaling free energy per unit volume for large constant fields, for the system perturbed from the fixed point:

$$V_p(\phi_p, 0) \sim \phi_p^{1+\delta} \left\{ A + ag_p\phi_p^{-1/\beta} \right\} \quad \text{for} \quad \phi_p \gg Cg_p^{\beta} \quad . \tag{4.3}$$

Here we have written $g_p = g\mu_p^{\lambda}$. The limit is finite for any finite g, as claimed, at least in this large ϕ_p region, and we see that it corresponds to a finite physical coupling g_p. Furthermore, if $V_p(\phi_p, 0)$ is finite for all ϕ_p, we have using (2.3), that

$$V(\phi, t) \sim V_p(\Lambda^d\phi, 0)/\Lambda^D \tag{4.4}$$

should satisfy (2.2) in the limit $\Lambda \to 0$. Substituting, we see that (2.2) is indeed satisfied, in fact up to corrections $\sim \ln(\Lambda)/\Lambda^D$.

We remark that higher order corrections to (4.3) are presumably directly calculable by continuing the perturbative expansion (4.1) to higher order in g, this being determined uniquely by similar considerations to sect.3.

Obviously the form of the boundary condition: (4.1), with (4.2) and $t \to -\infty$, is rather inconvenient, indeed is yet another limit, and the flow equation is numerically unstable (stiff) in such a region – since perturbations behave exponentially in t. We can resolve both these problems by expressing the continuum RG in terms of self-similar flow of the renormalised couplings (*i.e.* analogously to the field theory perturbative RG). In other words we rewrite in this case, (2.2) as a differential equation for $V(\phi, g)$ together with a (non-perturbative) beta function $\beta(g)$ for $g(t)$. The advantage of this is that the flow equation in g has perturbations that behave only as a power-law (around the fixed point) and, as we will see, has remarkably good numerical behaviour. At the same time we have resolved the limit since the boundary condition defining the RT, may now simply be stated as[12]

$$\frac{\partial}{\partial g} V(\phi, g) \Big|_{g=0} = v(\phi) \quad .$$

Rather than furnish further details at this stage, we combine this idea with another change of variables. Since there is really only one dimensionful parameter g_p,[13] it merely serves to fix the physical mass scale m_p. If we provide a physical definition for this mass-scale (equivalent to a renormalization scheme for g_p) then the dimensionless coupling constant ratios formed using powers of m_p, will be universal. Evidently these are generated by the dimensionless potential (and field):

$$U(\varphi) = V_p(m_p^d \varphi, 0)/m_p^D \quad , \tag{4.5}$$

from which it follows that the equation

$$U'(\varphi) = h \quad ,$$

is nothing but the universal scaling equation of state (h being the external 'magnetic field', equivalent to the source J in quantum field theory). Let us define a (dimensionless) running 'mass' $m \equiv m(g)$ with the property that it corresponds in the limit to the physical mass m_p, *i.e.*

$$m_p = \lim_{\Lambda \to 0} \Lambda\, m(g) \quad . \tag{4.6}$$

It will be helpful to change variables from g to m. Indeed, defining

$$U(\varphi, m) = V(m^d \varphi, t)/m^D \quad , \tag{4.7}$$

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[12] Similar remarks have been made in ref.[7].

[13]   corresponding to the one relevant direction, all μ_p dependence being absorbed in g_p

and using (2.3), (4.6) and (4.5), we see that $U(\varphi, m)$ has the property that it tends to the universal effective potential (4.5) in the limit $m \to \infty$. By rewriting the RG (2.2) as a flow equation for $U(\varphi, m)$, we obtain the universal physics directly as the large m behaviour. The flow with respect to t is never needed explicitly, having been 'hidden' in the beta-function for m: $\partial m/\partial t = \beta(m)$.

In the next three sections we flesh out the above sketch of changes of variables for the cases of LPA with sharp cutoff, LPA – equivalently $O(\partial^0)$ – for the smooth cutoff employed in [2], and $O(\partial^2)$ for this smooth cutoff. In all cases we study only the symmetric phase and use as definition of m_p,

$$m_p^2 \equiv V_p''(0,0) \quad , \tag{4.8}$$

in concord with refs.[14][25][17]. Then by (4.5), $U(\varphi)$ is normalised as

$$U''(0) = 1 \quad . \tag{4.9}$$

Again, the conclusions of this section are not particular to the LPA, and can be expected to hold also for the exact RG, with one proviso: beyond the LPA we would have to consider the effects also of wavefunction renormalisation. This is straightforwardly incorporated by a further change of variables (multiplicative renormalization) to a field with a kinetic term that remains correctly normalised under flow in t (or m). As a byproduct one obtains a non-perturbative expression for the running anomalous scaling $\gamma(m)$. We will provide the details in sect.7, where the $O(\partial^2)$ approximation will be developed.

5. Estimates from the LPA of the Wegner-Houghton RG

In this section we carry out explicitly the program outlined above, for equation (2.2), the result of the LPA applied to the RG for sharp cutoff. We will also explain our numerical methods for solving the resulting partial differential equations to obtain the universal coupling constant ratios *i.e.* the Taylor expansion coefficients of $U(\varphi)$. Since these results refer to the case of the Wilson-Fisher fixed point in three dimensions, we from now on set $D = 3$, implying for the LPA, $d = 1/2$.

Our first step is to define an appropriate running mass m. If we set

$$\sigma(t) = V''(0,t) \qquad \text{and (for later)} \qquad \hat{\alpha}(t) = V^{(4)}(0,t)/24 \quad , \tag{5.1}$$

then by (4.7) and (4.9),

$$\sigma/m^2 \to 1 \quad , \tag{5.2}$$

as both tend to infinity. We cannot simply have $m^2 = \sigma$ however, since σ starts out negative:

$$\sigma(t = -\infty) = \sigma_* = -.46153372 \cdots \tag{5.3}$$

[22][3], this being σ's fixed point value. Using this in eqn.(2.2), the Taylor expansion of $V(\phi)$ can be developed and hence of course fixed point values for $\hat{\alpha}$ and all the other couplings may be deduced. Some of these were quoted in ref.[28]. Since there has been confusion in some of the recent literature, it is worthwhile to stress that these numbers have nothing directly to do with the universal ratios in the scaling equation of state. In particular, fixed point values depend sensitively on the form of the cutoff; they are not universal.

Expanding (2.2) in powers of the field, and using (5.1), we have

$$\frac{\partial \sigma}{\partial t} = 2\sigma + \frac{24\hat{\alpha}}{1 + \sigma} \quad . \tag{5.4}$$

Clearly in general, on expanding (2.2), inverse powers of $1 + \sigma$ are generated, so to keep the algebra clean it is helpful to define

$$\sigma = -1 + m^2 \quad , \tag{5.5}$$

in agreement with (5.2). At the same time this ensures that m is real, since from (2.2) we clearly have $\sigma(t) > -1$ for all t. Now define the couplings in $U(\varphi, m)$ by,

$$U(\varphi, m) = \frac{1}{2} \frac{\sigma}{m^2} \varphi^2 + U_{int}(\varphi, m) \quad \text{with} \quad U_{int}(\varphi, m) = \mathcal{E} + \sum_{k=2}^{\infty} \varphi^{2k} \alpha_{2k}(m) \quad . \tag{5.6}$$

Thus, from (5.1) and (4.7) we have $\alpha_4 = \hat{\alpha}/m$. Differentiating (4.7) with respect to t, and using this relation, (5.4), (5.5) and (2.2), we obtain

$$\beta \frac{\partial}{\partial m} U(\varphi, m) = \left(\frac{1}{m^2} - \frac{12\alpha_4}{m^3} \right) \left(3U - \frac{1}{2} \varphi U' \right) + \frac{1}{m^3} \ln(1 + U''_{int}) \quad , \tag{5.7}$$

with the beta function,

$$\beta(m) = m - 1/m + 12\alpha_4(m)/m^2 \quad . \tag{5.8}$$

The φ^2 part of (5.7) is satisfied automatically by (5.5), while all other couplings, including α_4, are deduced as a function of m from (5.7) once the boundary condition at $m = m_* =$

$\sqrt{1+\sigma_*}$ is provided. This boundary condition may be deduced as follows. First we need to choose a normalization for $v(\phi)$ (cf. sect. 3). We choose,

$$v''(0) = 1 \quad , \tag{5.9}$$

which by (4.1) implies for small $g(t)$ (*i.e.* $t \to -\infty$), $\sigma = \sigma_* + g(t)$, and thus at the fixed point $\partial g(t)/\partial m = 2m_*$. Defining, analagously to (4.7),

$$U_*(\varphi) = V(\varphi\sqrt{m_*})/m_*^3 \qquad \text{and} \qquad u(\varphi) = v(\varphi\sqrt{m_*})/m_*^3 \quad , \tag{5.10}$$

substituting (4.1) into (4.7) and differentiating with respect to m gives the boundary condition for $\partial U/\partial m$,

$$\left.\frac{\partial U}{\partial m}\right|_* = \frac{1}{m_*}(\tfrac{1}{2}\varphi U_*' - 3U_*) + 2m_* u \quad . \tag{5.11}$$

Note that $\partial U/\partial m$ needs to be specified as well as U at the fixed point, because $\beta(m_*) = 0$ and the right hand side of (5.7) vanish there. [Thus $m = m_*$, $U = U_*$ corresponds to a so-called 'singular point' for the differential equation (5.7).]

Since our goal is the numerical computation of as many of the universal coupling constant ratios $\alpha_{2k}(\infty)$ [$k = 2, 3, \cdots$, *c.f.* eqn.(4.5) and below (4.7)] as possible, it makes sense to Taylor expand the differential equation (5.7) and boundary condition (5.11) in terms of φ and obtain directly ordinary differential flow equations for the $\alpha_{2k}(m)$. These are easily obtained to high order using algebraic computing packages. (We used Maple.) We can expect to obtain better estimates this way and with less effort compared to, say, numerically solving the partial differential equations for $U(\varphi, m)$, and then obtaining the $\alpha_{2k}(\infty)$ by numerically differentiating $U(\varphi, \infty)$, $2k$ times at $\varphi = 0$. But we now have a problem of truncation: the expansion of (5.7) leads to differential equations where the $\partial\alpha_{2k}/\partial m$ depend on all α_{2j} up to and including α_{2k+2}, and therefore if we keep only a finite number of these equations, say up to and including the equation $\partial\alpha_{2k}/\partial m = \cdots$, we will require an ansatz for the highest φ-power, the 'top coupling' $\alpha_{2k+2}(m)$. However, this problem is not as severe as for computations of fixed points by truncations in powers of the field[3], in particular since we already know the initial (fixed point) data numerically there will be no spurious solutions[3], and we can expect that the results converge as we keep more and more of the equations (and thus at the same time compute estimates for more and more of the couplings) providing we supply a decent ansatz for the top coupling. Of course if our ansatz corresponded exactly to the right answer for the top coupling,

all the lower couplings would be computed correctly. Our assumption is however that if a reasonable model of the top coupling's flow is inserted, then the lower couplings flow becomes more and more insensitive to the error in the modelling of the top coupling as k is increased (the error having to feed down successively through more equations). We find that the results bear out this assumption for several different ansätze, at least to sufficient accuracy for this exercise. These ansätze are outlined below. First we sketch the final steps necessary to obtain appropriate boundary conditions for these flow equations.

From the numerical value for σ_*, we deduce as many of the couplings in $V(\phi)$ as needed (*c.f.* (5.3) and below it), and thus from (5.5) and (5.10), as many of the initial values, $\alpha_{2k}(m_*)$, as needed. Similarly, by Taylor expansion of (3.2), using the normalization (5.9), and the relevant eigenvalue $\lambda = 1.450416(1)$ (already obtained numerically in the process of writing refs.[3][2]) we can obtain the Taylor series coefficients of $v(\phi)$. From (5.10) and (5.11), this then gives the numerical values for the initial gradients $\alpha'_{2k}(m_*)$. Some numerical methods can cope with this singular point [*c.f.* below (5.11)], but we found that the resulting differential equations were remarkably well-behaved numerically and that such sophistication was not necessary. Instead we used the basic fourth-fifth order Runge-Kutta provided within the Maple package, and set the boundary condition a little way away from the fixed point as $\alpha_{2k}(m_* + \delta) = \alpha_{2k}(m_*) + \alpha'_{2k}(m_*)\,\delta$. Here, δ could be set very small. We chose typically to set it to $\sim 10^{-6}$. The results were entirely insensitive to the precise choice. We integrated out to $m = 20$: with the improved estimate (5.13) described below this gave us more than sufficient accuracy while being easily achieved by the above package.

Turning now to the ansatz for the top coupling, the simplest workable ansatz of all, is to ignore its evolution and just set it equal to its initial value:

$$\alpha_{2k+2}(m) = \alpha_{2k+2}(m_*) \quad . \tag{5.12}$$

Numerically we found that this ansatz results in reasonable convergence, but it is easy to do better. From the above, we know also the initial gradient, while from the previous analysis (and borne out *e.g.* by the above ansatz) we expect that all couplings $\alpha_{2j}(m)$ 'run' with m initially and then freeze out at some value m_{freeze}, after which $\alpha_{2j}(m) \approx \alpha_{2j}(\infty)$. Indeed from (5.7) we see that asymptotically $\alpha_{2j}(m) \sim (1 - \frac{3-k}{2m^2})\alpha_{2j}(\infty)$, a fact which we used to improve the estimate of $\alpha_{2j}(\infty)$:

$$\alpha_{2j}(\infty) \sim \left(1 + \frac{3-k}{2m^2}\right)\alpha_{2j}(m) + O\left(\frac{1}{m^3}\right) \quad . \tag{5.13}$$

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This suggests the following simple ansatz for the top coupling,

$$\alpha_{2k+2}(m) = \alpha_{2k+2}(m_*) + \frac{\alpha_4(m) - \alpha_4(m_*)}{\alpha_4'(m_*)}\alpha_{2k+2}'(m_*) \quad , \tag{5.14}$$

which thus has the right value and gradient at $m = m_*$ and will freeze out at the same value m_{freeze} as α_4. Initially we employed an ansatz where α_4 was replaced in (5.14) by α_{2k}, in the expectation that any k dependence of m_{freeze} would thus be better taken into account:

$$\alpha_{2k+2}(m) = \alpha_{2k+2}(m_*) + \frac{\alpha_{2k}(m) - \alpha_{2k}(m_*)}{\alpha_{2k}'(m_*)}\alpha_{2k+2}'(m_*) \quad . \tag{5.15}$$

It is such an ansatz that led to the $O(\partial^2)$ estimates quoted in ref.[14]. However, we found that there was little or no drift of m_{freeze} with k, while using α_{2k} in place of α_4 in (5.14) led sometimes to numerical instabilities *e.g.* when $\alpha_{2k}'(m_*)$ happened to be much smaller than $\alpha_{2k+2}'(m_*)$, and to poorer convergence.

All three methods described led to consistent estimates for the $\alpha_{2k}(\infty)$. We present some of the raw data in sec.8. The most convergent answers were produced with ansatz (5.14). Using these, and untying the numerical transformation (2.4), *i.e.* as

$$\alpha_{2k}^{\mathrm{p}} = \zeta^{2k-2}\alpha_{2k}(\infty) \quad , \tag{5.16}$$

where by below (2.4), $\zeta = 2\pi$, yields our final results. These are given in table 7. They correspond to the normalization used in ref.[17].

6. Lowest order in the derivative expansion

In this section we carry out the program for the LPA of the RG written directly in terms of the Legendre effective action, *c.f.* (2.5), with respect to a smooth cutoff. This corresponds to $O(\partial^0)$ in a derivative expansion of Γ_Λ:

$$\Gamma_\Lambda[\phi] = \int d^3x \left\{ \tfrac{1}{2}(\partial_\mu \phi_{\mathrm{phys}})^2 + V_{\mathrm{phys}}(\phi_{\mathrm{phys}}, \Lambda) \right\} \quad , \tag{6.1}$$

with no other approximation made. (We drop the ^{c} superscript.) The relevant flow equations were derived in ref.[2], where details of the form of the cutoff *etc.* can be found. Since the critical exponents were found[2] to be better estimated by the smooth cutoff LPA compared to the sharp case described in the previous section, one might expect the universal couplings also to be better estimated. The results bear out these expectations.

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Instead of the flow equation (2.2), one finds[2]:

$$\frac{\partial}{\partial t}V(\phi,t) + \tfrac{1}{2}\phi V'(\phi,t) - 3V(\phi,t) = -1/\sqrt{2+V''(\phi,t)} \quad . \tag{6.2}$$

We define again $\sigma(t)$ and $\hat{\alpha}(t)$ by eqn.(5.1). However, Taylor expanding (6.2) one finds instead of (5.4),

$$\frac{\partial\sigma}{\partial t} = 2\sigma + 12\,\hat{\alpha}\,(2+\sigma)^{-3/2} \quad ,$$

the difference of course arising as a result of the inherently different 'blocking' procedure implied by use of the sharp cutoff. (Only universal quantities such as the exponents and the $\alpha_{2k}(\infty)$ are independent of the choice of cutoff and this only when the RG is computed exactly, otherwise some dependence on the choice of cutoff is left[5]. See also our comments below (5.3).) By similar reasoning to that for (5.5), a convenient definition for m is here,

$$\sigma = -2 + m^2 \quad . \tag{6.3}$$

Using the same expansion (5.6), we thus obtain the flow equation

$$\beta\frac{\partial}{\partial m}U(\varphi,m) = \left(\frac{2}{m^2} - \frac{6\alpha_4}{m^4}\right)\left(3U - \frac{1}{2}\varphi U\right) - \frac{1}{m^4\sqrt{1+U''_{int}}} \quad , \tag{6.4}$$

with the beta-function,

$$\beta(m) = m - 2/m + 6\alpha_4(m)/m^3 \quad .$$

Choosing the normalization (5.9) for the relevant perturbation, and using (4.1), (6.3) again implies $\partial g(t)/\partial m = 2m_*$ as $t \to -\infty$, and thus with definitions (5.10), the same formula (5.11) holds for the initial boundary conditions. Our treatment of these and the choice of ansatz for the top coupling are the same as in the previous section. From (6.4), the analogous formula to (5.13) reads however,

$$\alpha_{2j}(\infty) \sim \left(1 + \frac{3-k}{m^2}\right)\alpha_{2j}(m) + O\left(\frac{1}{m^4}\right) \quad . \tag{6.5}$$

The numerical values for the initial couplings follow from[2] $\sigma_* = -.534648257\cdots$, and the relevant eigenvalue $\lambda = 1.514260\cdots$. Eqn.(6.2) was also prettified by a numerical transformation of the form (2.4), and thus must be untied as in (5.16), but the corresponding value for ζ is $\zeta = \sqrt{2\pi}$ [2]. Numerical results for α_{2k}^{p} are given in sec.8.

7. Order derivative-squared

We use the smooth cutoff Legendre flow equation of the previous section, but improve the approximation by keeping also all terms of $O(\partial^2)$, *i.e.* we replace (6.1) with

$$\Gamma_\Lambda[\phi] = \int d^3x \left\{ \tfrac{1}{2}(\partial_\mu \phi_{\text{phys}})^2 K_{\text{phys}}(\phi_{\text{phys}}, \Lambda) + V_{\text{phys}}(\phi_{\text{phys}}, \Lambda) \right\} \quad . \qquad (7.1)$$

No further approximation is made. The resulting flow equations were derived in ref.[2]:

$$\frac{\partial V}{\partial t} + \frac{1}{2}(1+\eta)\phi V' - 3V = -\frac{1 - \eta/4}{\sqrt{K}\sqrt{V'' + 2\sqrt{K}}} \qquad (7.2a)$$

$$\text{and} \qquad \frac{\partial K}{\partial t} + \frac{1}{2}(1+\eta)\phi K' + \eta K = \left(1 - \frac{\eta}{4}\right) \left\{ \frac{1}{48} \frac{24KK'' - 19(K')^2}{K^{3/2}(V'' + 2\sqrt{K})^{3/2}} \right. \qquad (7.2b)$$

$$\left. - \frac{1}{48} \frac{58V'''K'\sqrt{K} + 57(K')^2 + (V''')^2 K}{K(V'' + 2\sqrt{K})^{5/2}} + \frac{5}{12} \frac{(V''')^2 K + 2V'''K'\sqrt{K} + (K')^2}{\sqrt{K}(V'' + 2\sqrt{K})^{7/2}} \right\} \quad .$$

The main novelty in the theoretical construction, compared to the previous sections, is the appearance of a non-zero anomalous dimension η. This is determined at the fixed point itself, as consequence of a ϕ rescaling invariance which is here preserved by the derivative expansion as a result of the careful choice of cutoff function. We refer the reader to refs.[2][5] for the details. They will not be important for the ensuing analysis. The non-zero value for η means that the couplings have anomalous scaling dimensions. These can be taken into account by approprate insertions of m^η, however, with an eye to future more general applications, we will choose the more conventional approach and introduce wavefunction renormalisation.

The change to dimensionless quantities is given, *c.f.* (2.3) and ref.[2], as

$$\phi = \phi_p \mu_p^{\eta/2}/\Lambda^d, \qquad V(\phi, t) = V_p(\phi_p, \Lambda)/\Lambda^3 \text{ and } K(\phi, t) = (\Lambda/\mu_p)^\eta K_p(\phi_p, \Lambda) \quad , \quad (7.3)$$

where now $d = \frac{1}{2}(1+\eta)$, and here the powers of μ_p are introduced to balance engineering dimensions. These physical quantities are related to the true physical quantities by the numerical transformation (2.4), and $K_p = K_{\text{phys}}$ [2], with $\zeta = \sqrt{2\pi}$ as in sec.6. As previously, we will correct for this transformation at the end of the section.

The physical fields do not yet have a normalised kinetic term. It is helpful to define

$$z(t) = 1/\sqrt{K(0,t)} \quad , \qquad (7.4)$$

then from (7.3) and (7.1), the renormalised physical fields

$$\phi_R = \mathrm{e}^{\eta t/2} \phi_p/z(t) \tag{7.5}$$

have a normalised kinetic term $K_R(\phi_R, \Lambda) = K_p(\phi_p, \Lambda)/K_p(0, \Lambda)$, satisfying $K_R(0, \Lambda) = 1$. (The normalisation of $V_R(\phi_R, \Lambda) = V_p(\phi_p, \Lambda)$ is of course unaffected.) Now we can define the physical mass scale [$c.f.$ (4.8)],

$$m_p^2 = V_R''(0,0) \quad, \tag{7.6}$$

and thus the universal dimensionless potential and $O(p^2)$ parts,

$$U(\varphi) = V_R(\varphi\sqrt{m_p},0)/m_p^3 \quad, \qquad L(\varphi) = K_R(\varphi\sqrt{m_p},0) \quad. \tag{7.7}$$

Note that $U(\varphi)$ is still normalised as in (4.9), while evidently L is normalised as

$$L(0) = 1 \quad. \tag{7.8}$$

Introducing again a running mass $m(t)$ with the property that it satisfies the limit (4.6), and combining the above transformations, we see that the quantities

$$U(\varphi, m) = V(\varphi z\sqrt{m}, t)/m^3 \qquad \text{and} \qquad L(\varphi, m) = z^2 K(\varphi z\sqrt{m}, t) \quad, \tag{7.9}$$

thoroughly enjoy the property that they tend to their universal counterparts as $m \to \infty$. Defining again $\sigma(t)$ as in (5.1), we see from (7.9) and (4.9), that (5.2) is replaced by the condition

$$\sigma z^2/m^2 \to 1 \quad. \tag{7.10}$$

Since $K(\phi, t)$ starts out normalised at the fixed point, $i.e.$ $K(0, -\infty) = K(0) = 1$ [2], we have by (7.4), the initial condition $z(t = -\infty) = z_* = 1$. On the other hand $\sigma(t = -\infty) = \sigma_*$ and we found numerically (in the process of writing ref.[2]), $\sigma_* = -.3781684\cdots$, so once again replacing the limit by equality in (7.10) will not do. By a similar analysis to that leading to (4.3) and (4.4), we may confirm that $K_p(\phi_p, \Lambda)$ has a finite limit as $\Lambda \to 0$, and thus determine from (7.3) that $z \sim m^{\eta/2}$ as $m \to \infty$. Since η is numerically small, $\eta = .05393208\cdots$ [2], the choice

$$m^2 = \sigma z^2 + 2z \tag{7.11}$$

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satisfies (7.10) while, being proportional to $\sigma + 2/z$, neatening up the small field expansion of eqns(7.2). Now, defining the couplings similarly to (5.6),

$$U(\varphi, m) = \frac{1}{2}\frac{\sigma z^2}{m^2}\varphi^2 + U_{int}(\varphi, m) \quad , U_{int}(\varphi, m) = \mathcal{E} + \sum_{k=2}^{\infty}\varphi^{2k}\alpha_{2k}(m) \quad , \quad (7.12a)$$

$$L(\varphi, m) = 1 + \sum_{k=1}^{\infty}\varphi^{2k}\gamma_{2k}(m) \quad , \quad (7.12b)$$

and substituting (7.9) into eqns(7.2), we obtain

$$\beta\frac{\partial}{\partial m}U(\varphi, m) = \frac{\beta - m}{m}\left(\frac{1}{2}\varphi U' - 3U\right) + \frac{\gamma - \eta}{2}\varphi U' - \left(1 - \frac{\eta}{4}\right)\frac{z^2}{m^4 R\sqrt{L}} \quad (7.13a)$$

$$\beta\frac{\partial}{\partial m}L(\varphi, m) = (\gamma - \eta)\left(\frac{1}{2}\varphi L' + L\right) + \frac{\beta - m}{2m}\varphi L' + \left(1 - \frac{\eta}{4}\right)\left\{\frac{z^2}{48}\frac{24LL'' - 19(L')^2}{m^4 R^3 L^{3/2}}\right. \quad (7.13b)$$

$$\left. - \frac{z}{48}\frac{58m^2 zU'''L'\sqrt{L} + 57z^2(L')^2 + m^4(U''')^2 L}{m^6 R^5 L} + \frac{5z^2}{12}\frac{m^4(U''')^2 L + 2m^2 zU'''L'\sqrt{L} + z^2(L')^2}{m^8 R^7 \sqrt{L}}\right\},$$

where we have introduced the short-hand,

$$R = \sqrt{1 + U''_{int} + \frac{2z}{m^2}\left(\sqrt{L} - 1\right)} \quad ,$$

and defined the beta function $\beta(m) = \partial m/\partial t$ and running anomalous dimension,

$$\gamma(m) = \frac{2}{z}\frac{\partial z}{\partial t} \quad .$$

The factor of 2 here corresponds to the usual definition in terms of a replacement $z = \sqrt{Z}$ in (7.4) and (7.5). These functions may be derived analagously to that of (5.4)–(5.8), or directly by evaluating the φ^2 coefficients in (7.13a), and the constant term in (7.13b), using (7.12). Thus,

$$\beta = m + \left(\frac{\eta}{2} - 2\right)\frac{z}{m} + \left(1 - \frac{\eta}{4}\right)\left(6\frac{z^2\alpha_4}{m^3} + \frac{z^3\gamma_2}{m^5}\right)$$

$$\gamma = \eta - \left(1 - \frac{\eta}{4}\right)\frac{z^2\gamma_2}{m^4} \quad . \quad (7.14)$$

To determine the initial boundary conditions we again use (4.1), together with the K component:

$$K(\phi, t) = K(\phi) + g(t)k(\phi) \quad , \quad (7.15)$$

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where $k(\phi)$ is the $O(\partial^2)$ part of the relevant operator [2]. It is simpler to implement the normalisation $k(0) = 1$, which thus replaces (5.9). If we define for convenience, $\tau = v''(0)$, then (4.1), (7.15), (5.1) and (7.4) imply that for $g(t)$ small, *i.e.* $t \to -\infty$,

$$\sigma(t) = \sigma_* + \tau g(t) \qquad \text{and} \qquad 1/z^2(t) = 1 + g(t) \quad .$$

Substituting (7.11) and differentiating with respect to m we deduce that at the fixed point,

$$\frac{\partial g}{\partial m} = -2 \frac{\partial z}{\partial m} = \frac{2m_*}{1 + \tau - m_*^2} \quad .$$

Thus, using the same definition (5.10) (because $z_* = 1$), and analogously

$$L_*(\varphi) = K(\varphi \sqrt{m_*}) \qquad , \qquad l(\varphi) = k(\varphi \sqrt{m_*}) \quad ,$$

combining (4.1), (7.15), and (7.9), and differentiating with respect to m gives the boundary conditions for the gradients,

$$\left. \frac{\partial U}{\partial m} \right|_* = \frac{1}{m_*} \left(\frac{1}{2} \varphi U' - 3U \right) + \frac{m_*}{m_*^2 - 1 - \tau} (\varphi U' - 2u)$$

$$\left. \frac{\partial L}{\partial m} \right|_* = \frac{1}{2m_*} \varphi L' + \frac{m_*}{m_*^2 - 1 - \tau} (2L + \varphi L' - 2l) \quad .$$

As before, we perform a small-field expansion of these equations, determining the initial data for the couplings in (7.12), in the way described in sec.5, using the numerical values of σ_* and η already given, and (from the analysis that led to ref.[2]) $\lambda = 1.61796660 \cdots$, $\tau = -24.472671 \cdots$.

Expanding (7.13) in powers of φ, one sees that the flow equations for the couplings have the following structure: two infinite sequences of equations, the differential equations for α_{2k}, with $k = 2, 3, \cdots$, and the γ_{2k} equations with $k = 1, 2, \cdots$, each of form,

$$\frac{\partial}{\partial m} \alpha_{2k}(m) = f_{2k}^{\alpha}(\alpha_4, \cdots, \alpha_{2k+2}; \gamma_2, \cdots, \gamma_{2k}) \quad ,$$

$$\frac{\partial}{\partial m} \gamma_{2k}(m) = f_{2k}^{\gamma}(\alpha_4, \cdots, \alpha_{2k+2}; \gamma_2, \cdots, \gamma_{2k+2}) \quad ,$$

(for some functions f_{2k}^{α}, f_{2k}^{γ}). This structure implies that, for given $k \geq 2$, there is not one unique minimal way to close the equations, rather there are two. These are:

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i) The differential equations for $\alpha_4, \cdots, \alpha_{2k}$ and $\gamma_2, \cdots, \gamma_{2k-2}$ are kept, with ansätze being supplied for γ_{2k} and α_{2k+2}.

ii) The differential equations for $\alpha_4, \cdots, \alpha_{2k}$ and $\gamma_2, \cdots, \gamma_{2k}$ are kept, with ansätze being supplied for γ_{2k+2} and α_{2k+2}.

Since there is no *a priori* reason to prefer one method over the other, we use both. Indeed, by employing at each k, first method (i) and then method (ii), we increment sequentially by one each time, the number of couplings which are estimated. For those top couplings that require an ansatz we continue to use (5.14), with also α_{2k+2} replaced by γ_{2k} or γ_{2k+2} as appropriate. (Of course other ansätze are possible.)

From a large m analysis of (7.13) and (7.14), we determine that an improved estimate for $\alpha_{2j}(\infty)$ can be obtained as

$$\alpha_{2j}(\infty) \sim \left(1 + (3-j)\frac{z(m)}{m^2}\right)\alpha_{2j}(m) \quad ,$$

which thus replaces (6.5). The γ_{2j} have also z/m^2 corrections but with a more complicated coefficient [the result of expansion of $(U''')^2(1 + U'')^{-5/2}$]. Rather than compute this coefficient algebraically, we determined it by a simple numerical fit on neighbouring values of m in the solution $\gamma_{2j}(m)$. This gave more than sufficient accuracy for the improved estimate.

Finally, we have to fold back in the missing numerical factors [*c.f.* below (7.3)], thus the final results are given as

$$\alpha_{2k}^{\mathrm{p}} = \zeta^{2k-2}\alpha_{2k}(\infty) \qquad \text{and} \qquad \gamma_{2k}^{\mathrm{p}} = \zeta^{2k}\gamma_{2k}(\infty) \quad , \tag{7.16}$$

where $\zeta = \sqrt{2\pi}$.

8. The numerical results

In this section we display the numerical results obtained from the analysis described in the previous three sections. In table 1, we compare the results obtained for α_4^{p}, at $O(\partial^0)$ approximation as in sec.6, employing the three different closure ansätze, (5.12), (5.14) and (5.15) for the 'top' coupling $\alpha_{2k+2}(m)$. One sees that indeed there is apparent convergence for each method as k is increased. We provide also in table 1 an estimate of the converged

answer for each ansatz, together with an error which reflects the spread of results over the several largest values of k in the table.

k	(5.12)	(5.15)	(5.14)
2	1.1955	1.3137	1.3137
3	1.2917	1.3097	1.3097
4	1.3117	1.3052	1.3041
5	1.3066	1.2954	1.3008
6	1.3004	1.3003	1.3003
7	1.2994	-	1.3010
8	1.3008	1.3017	1.3014
9	1.3017	1.3014	1.3014
10	1.3016	-	1.3012
∞	1.3016(8)	1.301(1)	1.3012(2)

Table 1. $O(\partial^0)$ results for α_4^{p}, at different levels k of truncation, using three different closure ansätze. For the two missing values in the middle column, the integrating routine failed to converge, due to wild numerical behaviour of the ansatz [*c.f.* below (5.15).]

The three methods give consistent estimates, although method (5.14) is seen to be the most powerful. Since we find that these facts hold true for the other couplings (and the other flow equations), from now on we present numerical results for method (5.14) only.

k	α_4^{p}	α_6^{p}	α_8^{p}	α_{10}^{p}	α_{12}^{p}	α_{14}^{p}	α_{16}^{p}	α_{18}^{p}	α_{20}^{p}
2	1.3137								
3	1.3097	2.7808							
4	1.3041	2.6873	1.964						
5	1.3008	2.6556	1.704	-3.566					
6	1.3003	2.6537	1.730	-2.718	-1.83				
7	1.3010	2.6600	1.779	-2.292	2.05	17.1			
8	1.3014	2.6632	1.795	-2.231	2.00	9.72	20.0		
9	1.3014	2.6628	1.790	-2.286	1.46	4.70	-24.8	-47.3	
10	1.3012	2.6615	1.783	-2.324	1.25	3.73	-26.4	-16.9	-389
∞	1.3012(2)	2.662(2)	1.78(1)	-2.3(1)	1.3(7)	6(4)	-20?	-20?	?

Table 2. $O(\partial^0)$ results for all computed couplings α_{2j}^{p}, at different levels k of truncation via (5.14), together with estimates for $k = \infty$.

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k	α_4^{p}	α_6^{p}	α_8^{p}	α_{10}^{p}	α_{12}^{p}
2 (i)	.8682				
2 (ii)	.8688				
3 (i)	.8644	1.253			
3 (ii)	.8667	1.269			
4 (i)	.8639	1.237	.586		
4 (ii)	.8646	1.240	.5790		
5 (i)	.8639	1.235	.5587	-0.950	
5 (ii)	.8633	1.230	.5275	-1.112	
6 (i)	.8635	1.232	.5475	-0.873	-.176
6 (ii)	.8630	1.229	.5359	-0.899	-.0688
∞	.8635(5)	1.230(3)	.54(1)	-.95(5)	?

Table 3. $O(\partial^2)$ estimates of α_{2j}^{p} at different levels of truncation k via (5.14), for the two closures described at the end of sec.7, together with estimates for $k = \infty$.

k	γ_2^{p}	γ_4^{p}	γ_6^{p}	γ_8^{p}	γ_{10}^{p}	γ_{12}^{p}
2 (i)	.6427					
2 (ii)	.6472	-3.023				
3 (i)	.6442	-3.047				
3 (ii)	.6537	-2.748	7.26			
4 (i)	.6504	-2.782	7.10			
4 (ii)	.6524	-2.771	6.59	7.82		
5 (i)	.6514	-2.776	6.55	7.3		
5 (ii)	.6499	-2.801	6.05	-2.5	-69	
6 (i)	.6502	-2.797	6.10	-2.20	-71.3	
6 (ii)	.6492	-2.805	6.02	-2.4	-53.7	-15
∞	.6497(5)	-2.805(5)	6.06(4)	-2.3(1)	-62(9)	?

Table 4. $O(\partial^2)$ estimates of γ_{2j}^{p} at different levels of truncation k via (5.14), for the two closures described at the end of sec.7, together with estimates for $k = \infty$.

The estimates resulting from LPA of the Wegner-Houghton equation are given in table 5. (The accuracy of these numbers was improved by correcting also for the $1/m^3$ contributions in (5.13), in a similar way to that done for γ_{2j}^{p} in the previous section.) To our knowledge these estimates for universal coupling constant ratios have not been derived

before.[14] (*N.B.* this approximation gives $\nu = .6895$ and $\omega = .5952$ [2][22].)

k	α_4^{P}	α_6^{P}	α_8^{P}	α_{10}^{P}	α_{12}^{P}	α_{14}^{P}	α_{16}^{P}	α_{18}^{P}
2	1.5308							
3	1.5288	3.569						
4	1.5184	3.439	2.142					
5	1.5107	3.377	1.833	-5.180				
6	1.5122	3.396	2.005	-3.491	5.88			
7	1.5163	3.426	2.142	-2.944	7.30	14.4		
8	1.5158	3.421	2.094	-3.317	4.48	-5.50	-58.6	
9	1.5135	3.404	2.027	-3.556	3.63	-6.49	-36.3	31
∞	1.514(2)	3.41(1)	2.08(6)	-3.3(3)	5(2)	-6?	-40?	?

Table 5. Sharp cutoff (Wegner-Houghton) with LPA, at different levels k of truncation via (5.14), together with estimates for $k = \infty$.

We can attempt to combine the $O(\partial^0)$ and $O(\partial^2)$ results of tables 2 and 3, to give numbers with an error which takes into account that due to truncation of the derivative expansion. Although two terms in the derivative expansion approximation are not really enough to take this exercise too seriously, we note that for all the results, including exponents[2],[15] the accuracy of the numbers obtained (compared to results from more accurate methods where these are available *e.g.* in ref.[14]), obey the following pattern: the sharp cutoff results are the worst, $O(\partial^0)$ are better, and $O(\partial^2)$ better still. Comparing the $O(\partial^0)$ and $O(\partial^2)$ estimates of exponents[16] ν and ω to the worlds best determinations we find that 1/3 of the difference between the $O(\partial^0)$ and $O(\partial^2)$ results give a good estimate of the error in the $O(\partial^2)$ results. If we adopt this algorithm for the couplings α_{2j}^{P} we obtain the numbers given in the next table. We include the data from ref.[2], and the most accurate estimates for the α_{2j}^{P} – three dimensional resummed perturbation theory – reported by ref.[14] [translated to this normalisation through eqn.(8.2)].

We have not included in table 6, the $O(\partial^2)$ estimate of η [see above (7.11)], or the $O(\partial^0)$ estimates of α_{12}^{P} and α_{14}^{P} given in table 2, because we have only one term in the derivative expansion for these and so cannot estimate the error due to truncation of the

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[14] We stress again that these do not correspond to exact RG fixed point couplings [*c.f.* our discussion above (5.4)].

[15] For the sharp cutoff case see above, and refs.[2][22]

[16] The $O(\partial^2)$ estimate for ω in table 6, corrects a misprint in ref.[2].

Approx'	ν	ω	α_4^{P}	α_6^{P}	α_8^{P}	α_{10}^{P}
$O(\partial^0)$	.6604	.6285	1.3012(2)	2.662(2)	1.78(1)	-2.3(1)
$O(\partial^2)$	.6181	.8972	.8635(5)	1.230(3)	.54(1)	-.95(5)
∂ exp$^{\mathrm{n}}$	.618(14)	.898(90)	.86(15)	1.2(5)	.5(4)	-1.0(5)
[29] & [14]	.631(2)	.80(4)	.988(2)	1.60(1)	.83(8)	-2.0(1.3)

Table 6. Estimates from the derivative expansion, with errors computed as described in the text, compared to combined results from the worlds best determinations for the exponents [29][2], and resummed perturbation theory for the α^{P}'s [14]. The line labelled "∂ exp$^{\mathrm{n}}$" gives a combined result from $O(\partial^0)$ and $O(\partial^2)$, as explained in the text above.

derivative expansion. Similarly, at $O(\partial^2)$ we obtain for the first time, estimates of γ_{2j}^{p}, as displayed in table 4. Note that the truncation level where we choose to stop in each of these tables, was determined by limitations of size and/or stability within the Maple computing package. It is certainly possible to do better, and thus estimate more higher order couplings, with a more serious attack *e.g.* by writing FORTRAN code, nor do we wish to rule out the possibility that a method based on direct integration of the partial flow equations derived in the previous sections, could produce competitive or better results than obtained here (while having the advantage of course of obtaining directly also the equation of state).

It can be seen from table 6, that while perturbative methods are more powerful than the derivative expansion for low order couplings, the derivative expansion eventually wins out. The reason for this is that the derivative expansion at these lowest orders, is crude in comparison to the perturbation theory methods, however the perturbative methods suffer from being asymptotic – which in particular results in rapidly worse determinations for higher order couplings. The derivative expansion does not suffer from this, since it is not related at all to an expansion in powers of the field. Indeed, it may be shown that[6] even at the level of the LPA, Feynman diagrams of all topologies are included. We recall here that derivative expansion estimates can successfully be given for the multicritical fixed points in two dimensions[9], where all other standard methods fail.[17] In that study, we found that derivative expansion estimates actually tend to improve for higher dimension corrections

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[17] Needless to say, it is possible, and would be very interesting, to apply the methods developed here to the two-dimensional cases and thus derive scaling equations of state for each of these multicritical points.

to scaling (*i.e.* higher dimension operators) and for greater multicriticality. This trend is opposite to that in perturbation theory (or any other standard approximation method for that matter). This suggests that the main source of error in the derivative expansion estimates of the α_{2j}^{p}'s is that of the α_4^{p} coupling. We follow Guida and Zinn-Justin [14] and factor out the α_4^{p} coupling,[18] by defining

$$
\begin{aligned}
f(z) &= 24\,\alpha_4(\infty)\left\{ U\left(z/\sqrt{24\,\alpha_4(\infty)}\right) - \mathcal{E}\right\} \\
&= z^2/2 + z^4/4! + \sum_{l=3} z^{2l} F_{2l-1}/2l \quad,
\end{aligned}
\tag{8.1}
$$

c.f. (5.6), (4.5) and (4.7). The F's are then given by

$$
F_{2l-1} = 2l\,\alpha_{2l}(\infty)/\left[24\,\alpha_4(\infty)\right]^{l-1} = 2l\,\alpha_{2l}^{\mathrm{p}}/(24\,\alpha_4^{\mathrm{p}})^{l-1} \quad.
\tag{8.2}
$$

Note that they are independent of the numerical normalisation factors (7.16). We find much improved derivative expansion estimates of the higher order F's compared to the α^{p}'s. This is illustrated in table 7. It seems therefore that the main source of error in the derivative expansion can be absorbed in an effective normalisation factor $\zeta_{eff} = \zeta\sqrt{(\alpha_4^{\mathrm{p}})_{exact}/\alpha_4^{\mathrm{p}}}$. At $O(\partial^0)$, $\zeta_{eff} = 1.148(1)\,\zeta$. At $O(\partial^2)$, $\zeta_{eff} = .935(1)\,\zeta$. We have taken the opportunity to include in table 7, the estimates from the sharp cutoff LPA data in table 5, and comparisons with estimates from other methods.[19] We use the same heuristic as before to combine the $O(\partial^0)$ and $O(\partial^2)$ results into a single derivative expansion estimate with error. In addition, at $O(\partial^0)$ we obtain the estimates $F_{11} = 5(3) \times 10^{-7}$, $F_{13} = 9(6) \times 10^{-8}$, and with sharp cutoff LPA, the estimate $F_{11} = 9.5(4.0) \times 10^{-7}$ (consistent with the $O(\partial^0)$ result). To our knowledge there are no other estimates with which these can be compared.

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[18] Ref.[14]'s coupling $g^* \equiv 24\,\alpha_4^{\mathrm{p}}$.

[19] We recall again that the $O(\partial^2)$ estimates quoted in the corresponding table in ref.[14] were derived using the less powerful closure ansatz (5.15). They are consistent with the present estimates but with a larger error – reflecting the poorer convergence with truncation level k. For ref.[17], we list only their more sophisticated Padé-Borel estimates.

Approx'	α_4^{p}	F_5	F_7	F_9
Sharp	1.514(2)	.0155(3)	$3.6(5) \times 10^{-4}$	$-1.7(5) \times 10^{-5}$
$O(\partial^0)$	1.3012(2)	.01638(1)	$4.68(3) \times 10^{-4}$	$-2.4(1) \times 10^{-5}$
$O(\partial^2)$	.8635(5)	.01719(4)	$4.9(1) \times 10^{-4}$	$-5.2(3) \times 10^{-5}$
∂ exp$^{\mathrm{n}}$	.86(15)	.0172(3)	$4.9(1) \times 10^{-4}$	$-5(1) \times 10^{-5}$
$d = 3$ [14]	.988(2)	.01712(6)	$4.96(49) \times 10^{-4}$	$-6(4) \times 10^{-5}$
$d = 3$ [17]		.0168 - .0173	4.1 - 16.2 $\times 10^{-4}$	
$\varepsilon-$exp. [14]	1.2	.0176(4)	$4.5(3) \times 10^{-4}$	$-3.2(2) \times 10^{-5}$
$\varepsilon-$exp. [17]		.0176		
ERG [25]	1.20	.016	4.3×10^{-4}	
HT [30]	.99(6)	.0205(52)		
HT [31]	1.019(6)	.01780(15)		
HT [32]	.987(4)	.017(1)	$5.4(6) \times 10^{-4}$	$-2(1) \times 10^{-5}$
MC [16]	.97(2)	.0227(26)		
MC [33]	1.020(8)	.027(2)	.00236(40)	

Table 7. Estimates for α_4^{p} and the F_{2l-1}: from the derivative expansion – with errors computed as described in the text, the LPA of the Wegner-Houghton equation, and (summaries of) other methods. Again the line labelled "∂ exp$^{\mathrm{n}}$" gives combined results from $O(\partial^0)$ and $O(\partial^2)$, in a way described earlier in the text.

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